MATH 590: QUIZ 9 SOLUTIONS

Name:

1. Let V denote the vector space of real polynomials having degree less than or equal to two, with inner product $\langle f(x), g(x)\rangle := \int_{-1}^{1} f(x)g(x)\,dx$. Consider the orthonormal basis for V given by $f_1 := \frac{1}{\sqrt{2}}$, $f_2 := \sqrt{\frac{3}{2}}x$ and $f_3 := \sqrt{\frac{5}{8}}(3x^2-1)$. Verify that $\langle f_2, f_3\rangle = 0$ and $||f_2|| = 1$. (5 points).

Solution For the first statement,

$$\begin{aligned}\langle f_2, f_3\rangle &= \int_{-1}^{1} \sqrt{\frac{3}{2}}x \cdot \sqrt{\frac{5}{8}}(3x^2-1)\ dx\\ &= \sqrt{\frac{15}{16}}\int_{-1}^{1} 3x^3 - x\ d\\ &= \sqrt{\frac{15}{16}}\left\{\frac{3}{4}x^4 - \frac{1}{2}x^2\right\}_{-1}^{1} = \sqrt{\frac{15}{16}}\left\{(\frac{3}{4}-\frac{1}{2}) - (\frac{3}{4}-\frac{1}{2})\right\} = 0\end{aligned}$$

For the second statement,

$$||f_2|| = \left\{\int_{-1}^{1}\sqrt{\frac{3}{2}}x\cdot\sqrt{\frac{3}{2}}x\ dx\right\}^{\frac{1}{2}} = \left\{\frac{3}{2}\int_{-1}^{1} x^2\ dx\right\}^{\frac{1}{2}} = \left\{\frac{3}{2}\cdot(\frac{x^3}{3})_{-1}^{1}\right\}^{\frac{1}{2}} = \left\{\frac{3}{2}\cdot\frac{2}{3}\right\}^{\frac{1}{2}} = 1.$$

2. Let $W \subseteq \mathbb{R}^3$ be the subspace spanned by $v_1 := (1,2,1)$ and $v_2 := (-1,4,0)$. Find an orthonormal basis for W. (5 points)

Solution. We first find an orthogonal basis $\{w_1, w_2\}$ for W via Gram-Schmidt. Take $w_1 = v_1$. Take

$$w_2 = v_2 - \frac{v_2\cdot w_1}{w_1\cdot w_1}w_1 = (-1,4,0) - \frac{7}{6}(1,2,1) = (-\frac{13}{6}, \frac{10}{6}, -\frac{7}{6}).$$

$||w_1|| = \sqrt{6}$ and $||w_2|| = \sqrt{\frac{318}{36}} = \sqrt{\frac{53}{6}}$. Thus, $u_1 = \frac{1}{\sqrt{6}}(1,2,1)$ and $u_2 = \sqrt{\frac{6}{53}}(-\frac{13}{6}, \frac{10}{6}, -\frac{7}{6})$ is an orthonormal basis for W.